RESEARCH NOTE

A Note on the Estimation of Mean Rate of Change*

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GIVEN a set of data represented by n points $P_i\ (x_i,\ y_i)$ on ordinary rectangular graph paper with the x_i not necessarily regularly spaced, the determination of the mean rate of change (*m.r.c.*) of the dependent variable y is often a problem of practical importance. The standard procedure is to apply the method of least squares and calculate the *m.r.c.* from the value of the linear regression coefficient,

$$b = \frac{\sum_{i=1}^{n} (x_i - \bar{x})\ (y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}. \tag{1}$$

Askovitz (1956) has given an interesting alternative method for estimating *m.r.c.* It consists in joining on the graph, pairs of points P_i and P_j in all possible ways and assigning to each slope:

$$m_{ij} = \frac{y_j - y_i}{x_j - x_i}\ (1 \leqslant i < j \leqslant n) \tag{2}$$

a weight $w_{ij} = x_j - x_i$. Since the x's are arranged in ascending order of magnitude the weights are positive.

The weighted average of the slopes of all these segments provides the following formula for *m.r.c.*

$$\begin{aligned} b' &= \frac{\Sigma\ \Sigma\ w_{ij}\, m_{ij}}{\Sigma\ \Sigma\ w_{ij}}\ (1 \leqslant i < j \leqslant n) \\ &= \frac{\Sigma\ \Sigma\ (y_j - y_i)}{\Sigma\ \Sigma\ (x_j - x_i)}\ (1 \leqslant i < j \leqslant n) \\ &= \frac{\Sigma\ (2i - n - 1)\ y_i}{\Sigma\ (2i - n - 1)\ x_i}\ (i = 1,\ 2,\ \cdots\ n). \end{aligned} \tag{3}$$

From certain symmetrical properties of unbiased estimates of variance and covariance proved by the present author (1948) we can show that the least square estimate given in formula (1) is

$$b = \frac{\Sigma\ \Sigma\ (x_j - x_i)\ (y_j - y_i)}{\Sigma\ \Sigma\ (x_j - x_i)^2}\ (1 \leqslant i < j \leqslant n). \tag{4}$$

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This is a weighted average of m_{ij}, where the weight $w_{ij} = (x_j - x_i)^2$.

It is easy to prove that b and b' become identical when the x's are equally spaced. In that case a simple method of estimating the *m.r.c.* given by Nair and Shrivastava (1942) is to be commended. When y is normally distributed about the regression line of y on x their method has a relative efficiency of over 88 per cent. compared to the least square estimate.

It should be interesting to calculate the relative efficiency of b' compared to b when x's are not equally spaced.

From equation (42) of the author's (1948) paper it can be seen that the denominator of the last of the expressions on the right-hand side of (3) above is related to Gini's mean difference of x which can be written as

$$g_x = \frac{2}{n(n-1)} \Sigma\Sigma (x_j - x_i) \; (1 \leqslant i < j \leqslant n) \text{ and } (x_i < x_j)$$

$$= \frac{2}{n(n-1)} \sum_{i=1}^{n} \{(2i - n - 1) x_i\} \tag{5}$$

The variance of b' is

$$V(b') = \frac{\sum_{i=1}^{n} (2i - n - 1)^2}{\{\sum_{i=1}^{n} (2i - n - 1) x_i\}^2} \cdot \sigma^2_{y.x}$$

$$= \frac{4(n+1)}{3n(n-1)} \frac{\sigma^2_{y.x}}{g^2_x} \tag{6}$$

where $\sigma_{y.x}$ is the residual standard deviation of y about the regression line of y on x.

The variance of b is given by the well-known formula

$$V(b) = \frac{\sigma^2_{y.x}}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{\sigma^2_{y.x}}{(n-1) S_x^2} \tag{7}$$

where S_x^2 is the observed variance of x.

The relative efficiency of b' compared to b as alternative estimates of *m.r.c.* is given by

$$\frac{V(b)}{V(b')} = \frac{3n}{4(n+1)} \frac{g_x^2}{S_x^2} \tag{8}$$

This relative efficiency should obviously be $\leqslant 1$. In other words, we can conclude that

$$\frac{g_x}{S_x} \leqslant \sqrt{\frac{4(n+1)}{3n}}. \tag{9}$$

This inequality could, however, be proved independently. For instance, see last paragraph of the author's (1948) paper where it is shown that the limiting equality relation in (9) is reached only when the x's are equally spaced in which case, as already stated above, b and b' become identical.

We should be more interested in finding the *lower* bound of g_x/S_x, in terms of n if one exists, since that would enable us to calculate for a given n the lower limit of the relative efficiency of Askovitz's method of estimating *m.r.c.* compared to the least square method. The present author has not succeeded in finding this out.

Finally, as Askovitz points out, the symmetry in the x's and y's in the expression for b' should simplify its numerical evaluation. Also, he gives an interesting graphic method for evaluating b'.

References

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